SOME CASES OF UNSTEADY MOTION OF VISCOPLASTIC MEDIA IN AN INFINITELY LONG VISCOELASTIC TUBE

R. M. Sattarov

UDC 532.595.2+135

§ 1.We consider the unsteady motion of a compressible viscoplastic medium, whose properties vary in accordance with power and nonlinear laws, in a thin-walled tube of a viscoelastic Maxwellian material. The motion is represented by the equation $\dot{\gamma} = \dot{\tau}/G + \tau/\mu$.

We assume that initially the flow and pressure in the whole tube, occupying the half-space $x \geq 0$, is constant and equal to zero and at time $t = 0$ a pressure $p = \varphi(t)$ or a flow rate $M = \psi(t)$ is imparted at the boundary $x = 0$.

The solution of the problem reduces to solution of the system of differential equations [1]

$$\frac{1}{f_0}\frac{\partial M}{\partial t} = -\frac{\partial p}{\partial x} - m_j M; \tag{1.1}$$

$$\frac{R}{\delta}\frac{\partial p}{\partial t} + \left(\frac{R}{\delta}\frac{\mu}{G} + \frac{\mu}{2K_F}\right)\frac{\partial^2 p}{\partial t^2} + \frac{\mu}{2\rho_0 f_0}\frac{\partial^2 M}{\partial t\,\partial x} = 0, \tag{1.2}$$

$$m_1 = \frac{16\eta_a}{4\rho_0 f_0 R r}\left(\frac{3n+1}{4n}\right), \quad m_2 = \frac{16\eta_p}{4\rho_0 f_0 R r}\frac{1}{\sigma},$$

$$\sigma = \frac{\left(1 - \beta_0^{1/n}\right)^{n-1}}{C^n_{4n-1}}\sum_{k=0}^{3n-1} C^n_{4n-k-1}\beta_0^{k/n}.$$

When $j = 1$, Eq. (1.1) describes the motion of a "power-law" medium, and when $j = 2$, Eq. (1.1) represents the motion of a nonlinear viscoplastic medium. Here p is the pressure, M is the mass flow, R is the tube radius, δ is the thickness of the tube walls, ρ_0 is the density of the medium, f_0 is the cross-sectional area of the tube, K_F is the modulus of elasticity of the fluid, G is the shear modulus of the tube material, μ is the viscosity of the tube material, η_a is the apparent viscosity, η_p is the analog of plastic viscosity, r is the hydraulic radius, n is the nonlinearity parameter, β_0 is the dimensionless radius of the core, σ is the shear stress of the tube material, and γ is the shear strain of the tube material; the dot above the letters denotes the operator d/dt. The more accurate value of m_2 was obtained from [2], since in [1] it was determined for small values of the flow core. The initial and boundary conditions for the given problem have the form

$$M(x,t) = 0,\ p(x,t) = 0,\ t \leqslant 0;$$

$$\left.\begin{array}{l} p(0,t) = \varphi(t) \\ M(0,t) = \psi(t) \end{array}\right\} t > 0 \quad \begin{array}{l} \text{case A}, \\ \text{case B}. \end{array}$$

In addition, we assume that functions p(x, t) and M(x, t) are bounded as $x \to \infty$.

We solve the problem for case A. Eliminating the mass flow from Eqs. (1.1) and (1.2), we obtain a differential equation for p

$$A\partial^3 p/\partial t^3 + B\partial^2 p/\partial t^2 + D\partial p/\partial t = \partial^3 p/\partial t\partial x^2 \tag{1.3}$$

with initial and boundary conditions

$$\begin{cases} p(0,x) = 0, \quad \partial p(0,x)/\partial t = 0, \quad \partial^2 p(0,x)/\partial t^2 = 0, \\ p(t,0) = \varphi(t), \quad p(t,\infty) = 0, \end{cases} \tag{1.4}$$

Baku. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 93-99, May-June, 1977. Original article submitted May 31, 1976.

where

$$A = R/\delta \cdot 2\rho_0/G + \rho_0/K_F = 1/C^2; \quad B = f_0 m_j/C^2 + 2Rf_0/\delta\mu; \quad D = 2R\rho_0 f_0 m_j/\delta\mu.$$

Applying the Laplace transform to (1.3) and (1.4), we obtain

$$\partial^2 p^*/\partial x^2 - (As^2 + Bs + D)p^* = 0; \tag{1.5}$$

$$p^*(s, 0) = \varphi^*(s), \; p^*(s, \infty) = 0. \tag{1.6}$$

$$p^*(s, x) = \int_0^\infty e^{-st} p(x, t)\, dt, \quad \varphi^*(s) = \int_0^\infty e^{-st}\varphi(t)\, dt.$$

The solution of (1.5) with boundary conditions (1.6) has the form

$$p^*(s, x) = \varphi^*(s) \exp(-x\sqrt{As^2 + Bs + D}).$$

Converting to the original, we have

$$p(x, t) = \begin{cases} 0 & \text{for} \quad 0 \leqslant t < \frac{x}{C} \\ e^{-\frac{B}{2}Cx}\varphi\left(t - \frac{x}{C}\right) - xC\sqrt{\beta}\int\limits_{\frac{x}{C}}^{t} \varphi(t - \tau)\, e^{-\frac{B}{2}C^2\tau} \frac{J_1\left(C^2\sqrt{\beta}\sqrt{\tau^2 - \frac{x^2}{C^2}}\right)}{\sqrt{\tau^2 - \frac{x^2}{C^2}}}\, d\tau & \text{for} \quad t \geqslant \frac{x}{C}, \end{cases} \tag{1.7}$$

where $\beta = AD - (B/2)^2$.

The mass flow is given by the formula

$$M = -f_0 \int_0^t e^{-f_0 m_j(t-\tau)} \frac{dp(x, \tau)}{dx}\, d\tau. \tag{1.8}$$

By a similar procedure we obtain the solution for case B in the form

$$M(x, t) = \begin{cases} 0 & \text{for} \quad 0 \leqslant t < \frac{x}{C} \\ e^{-\frac{B}{2}Cx}\psi\left(t - \frac{x}{C}\right) - xC\sqrt{\beta}\int\limits_{\frac{x}{C}}^{t} \psi(t - \tau)\, e^{-\frac{B}{2}C^2\tau} \frac{J_1\left(C^2\sqrt{\beta}\sqrt{\tau^2 - \frac{x^2}{C^2}}\right)}{\sqrt{\tau^2 - \frac{x^2}{C^2}}}\, d\tau & \text{for} \quad t \geqslant \frac{x}{C}; \end{cases} \tag{1.9}$$

$$p(x, t) = -\int_0^x \left(\frac{1}{f_0}\frac{\partial M}{\partial t} + m_j M\right) dx. \tag{1.10}$$

The variation of the tube cross section f is given by the expression [1]

$$f - f_0 = (2f_0/\delta\mu)(\partial p/\partial t + t_0 p),$$

where $t_0 = \mu/G$ is the relaxation time.

The quantities B and D contained in formulas (1.7)-(1.10) are proportional to m_j and inversely proportional to μ. The parameter m_1 ("power-law" fluid) varies with change in n, other conditions being equal, in the range $3\eta_a/\rho_0 f_0 Rr - \infty$; an increase in n(n > 1 – dilatant medium) leads to a reduction of m_1 and, hence, B and D, while a reduction of n(n < 1 – pseudoplastic medium) leads to an increase in m_1, B, and D. An analysis of the parameter m_2 shows that an increase in the nonlinearity parameter, or in the yield stress $\tau_0(\beta_0)$ increases the values of m_2, B, and D.

The foregoing shows that in the motion of a "power-law" medium reduction of pressure and flow with time is more rapid for pseudoplastic media than for viscous and dilatant media. For a nonlinear viscoplastic medium the increase in the nonlinearity parameter n and the yield stress $\tau_0(\beta_0)$ leads to more rapid reduction of pressure and flow.

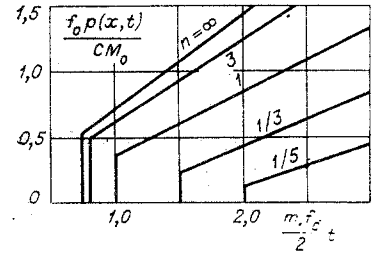

Fig. 1

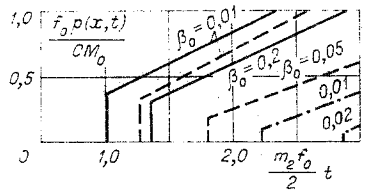

Fig. 2

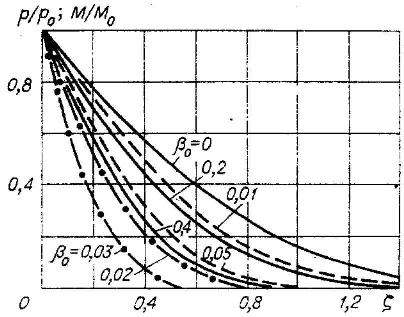

Fig. 3

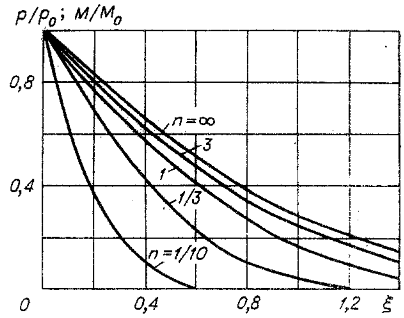

Fig. 4

Reduction of the viscosity of the tube material (which is equivalent to reduction of the relaxation time of the tube material) also leads to reduction of pressure and flow.

The foregoing expresses the important idea that the motion can be affected by altering either the characteristics of the medium being pumped or the viscosity of the tube material.

If we let $\mu \to \infty$ in Eqs. (1.7)-(1.10), they lead to expressions for the pressure and flow in an elastic tube, and in this case

$$B = f_0 m_j / C^2;\ D = 0.$$

Let us now assign a pressure or flow jump at the start of the elastic tube, i.e.,

$$\varphi(t) = \begin{cases} 0 & t < 0 \\ p_0 & t \geqslant 0 \end{cases} \quad \text{at} \quad \psi(t) = \begin{cases} 0 & t < 0 \\ M_0 & t \geqslant 0. \end{cases} \tag{1.11}$$

Substituting the boundary conditions (1.11) in Eqs. (1.7)-(1.10) we obtain in case A

$$p(x, t) = p_0 \begin{cases} 0 & \text{for} \quad 0 \leqslant t < \frac{x}{C} \\ e^{-\frac{f_0 m_j}{2C} x} + \frac{f_0 m_j}{2C} x \int\limits_{\frac{x}{C}}^{t} e^{-\frac{f_0 m_j}{2}\tau} \frac{I_1\left(\frac{f_0 m_j}{2} \sqrt{\tau^2 - \frac{x^2}{C^2}}\right)}{\sqrt{\tau^2 - \frac{x^2}{C^2}}} d\tau & \text{for} \quad t \geqslant \frac{x}{C}; \end{cases}$$

$$M(x, t) = \frac{p_0 f_0}{C} \begin{cases} 0 & \text{for} \quad 0 \leqslant t < \frac{x}{C} \\ e^{-\frac{f_0 m_j}{2} t} I_0\left(\frac{f_0 m_j}{2} \sqrt{t^2 - \frac{x^2}{C^2}}\right) & \text{for} \quad t \geqslant \frac{x}{C}; \end{cases}$$

in case B

$$p(x, t) = \frac{M_0 C}{f_0} \begin{cases} 0 & \text{for} \quad 0 \leqslant t < \frac{x}{C} \\ e^{-\frac{f_0 m_j}{2C} x} + \frac{m_j f_0}{2} \int\limits_{\frac{x}{C}}^{t} e^{-\frac{f_0 m_j}{2}\tau} \left[I_0\left(\frac{f_0 m_j}{2} \sqrt{\tau^2 - \frac{x^2}{C^2}}\right) + \right. \\ \left. + \frac{\tau I_1\left(\frac{f_0 m_j}{2} \sqrt{\tau^2 - \frac{x^2}{C^2}}\right)}{\sqrt{\tau^2 - \frac{x^2}{C^2}}} \right] d\tau & \text{for} \quad t \geqslant \frac{x}{C}; \end{cases} \tag{1.12}$$

$$M(x,t) = M_0 \begin{cases} 0 & \text{for} \quad 0 \leqslant t < \frac{x}{C} \\ e^{-\frac{f_0 m_j}{2C}x} + \frac{f_0 m_j}{2C} x \int\limits_{\frac{x}{C}}^{t} e^{-\frac{f_0 m_j}{2}\tau} \frac{I_1\left(\frac{f_0 m_j}{2}\sqrt{\tau^2 - \frac{x^2}{C^2}}\right)}{\sqrt{\tau^2 - \frac{x^2}{C^2}}} d\tau & \text{for} \quad t \geqslant \frac{x}{C}. \end{cases}$$

Figures 1 and 2 give the results of calculations from Eq. (1.12) for the motion of a "power-law" medium and a nonlinear viscoplastic medium, respectively. In this case for the "power-law" medium we took $m_{10}x/C = 1$ and for the nonlinear viscoplastic medium, $m_{20}x/C = 1$, where m_{10} and m_{20} have the form

$$m_{10} = 4\eta_a/\rho_0 R^2; \; m_{20} = 4\eta_p/\rho_0 R^2.$$

In Figs. 2 and 3 the solid lines correspond to the parameter n = 1; the dashed lines, to n = 2; and the dashed-dot lines, to n = 3.

Figure 1 shows that reduction of the rheological parameter n of a "power-law" medium increases the time lag in propagation of the jump. As $n \to 0$ the lag tends to infinity, i.e., the pressure over the whole tube is constant and equal to zero. This is probably due more to the known limitations of the "power-law" rheological equation in the range of small shear rates than to the actual behavior of the medium in the tube.

Figure 2 shows that an increase in the parameter n and radius of the quasicore zone leads to an increase in the time lag. With increase in the nonlinearity parameter n the increase in the time lag becomes more rapid.

§ 2. We consider the unsteady motion of a compressible "power-law" medium and a nonlinear viscoplastic medium in a thin-walled tube of viscoelastic material. This motion is represented by the Voigt equation $\tau = G\gamma + \mu\dot{\gamma}$. The formulation of the problem is the same as in Sec. 1.

The solution of the problem reduces to solution of the system of differential equations [1]

$$\frac{1}{f_0}\frac{\partial M}{\partial t} + m_j M = -\frac{\partial p}{\partial x}; \tag{2.1}$$

$$\left(\frac{R}{\delta} + \frac{G}{2K_F}\right)\frac{\partial p}{\partial t} + \frac{G}{2\rho_0 f_0}\frac{\partial M}{\partial x} + \frac{\mu}{2K_F}\frac{\partial^2 p}{\partial t^2} + \frac{\mu}{2\rho_0 f_0}\frac{\partial^2 M}{\partial t \partial x} = 0 \tag{2.2}$$

(the notation is the same as in Sec. 1).

The initial and boundary conditions for this problem have the form

$$M(x,\ t) = 0,\ p(x,\ t) = 0,\ t \leqslant 0;$$
$$\left.\begin{aligned} p(0,t) &= \varphi(t) \\ M(0,t) &= \psi(t) \end{aligned}\right\} t > 0 \quad \begin{aligned} &\text{case} \quad A, \\ &\text{case} \quad B. \end{aligned}$$

Functions p(x, t) and M(x, t) are bounded as $x \to \infty$. The problem is first solved for case A. Eliminating the flow M from Eqs. (2.1) and (2.2), we obtain a differential equation for p

$$A\frac{\partial^3 p}{\partial t^3} + B\frac{\partial^2 p}{\partial t^2} + D\frac{\partial p}{\partial t} = \lambda\frac{\partial^2 p}{\partial x^2} + \frac{\partial^3 p}{\partial t \partial x^2} \tag{2.3}$$

with initial and boundary conditions

$$\begin{cases} p(0,x) = 0, \quad \frac{\partial p(0,x)}{\partial t} = 0, \quad \frac{\partial^2 p(0,x)}{\partial t^2} = 0, \\ p(0,t) = \varphi(t), \quad p(t,\infty) = 0, \end{cases} \tag{2.4}$$

where $A = \rho_0/K_F$; $B = \lambda/C^2 + Am_j f_0$; $D = f_0\lambda m_j/C^2$; $1/\lambda = \mu/G$ is the time lag. Applying the Laplace transform to the system of equations (2.3) and (2.4) and solving the obtained differential equation for the images of the function p(x, t), we obtain

$$p^*(s,x) = \varphi^*(s)\exp\left(-x\sqrt{\frac{As^3 + Bs^2 + Ds}{s+\lambda}}\right). \tag{2.5}$$

The original of (2.5) can be obtained in the same way as in [3]. Below we analyze only the asymptotic behavior of the solution, however, since the general solution leads to rather unwieldy expressions.

When $s\mu/G \gg 1$, expression (2.5) in the original has the same form as Eq. (1.7) and, hence, expressions for the flow and the solution for case B can be determined from Eqs. (1.8)-(1.10), respectively. All the conclusions made in Sec. 1 are obviously still meaningful for this case too. If $s\mu/G \ll 1$, the original of (2.5) has the form

$$p(x, t)=\frac{2}{\sqrt{\pi}} \int_{\frac{x\sqrt{f_0 m_j}}{C\sqrt{t}}}^{\infty} \varphi\left(t-\frac{x\sqrt{f_0 m_j}}{4Cz^2}\right) e^{-z^2} dz. \tag{2.6}$$

The change in cross section f is given by the expression [1]

$$f-f_0=\frac{2f_0R}{\delta\mu}\int_0^t e^{-\lambda(t-\tau)} p\,d\tau.$$

The flow can be determined from Eq. (1.8). For case B we have

$$M(x, t)=\frac{2}{\sqrt{\pi}} \int_{\frac{x\sqrt{f_0 m_j}}{C\sqrt{t}}}^{\infty} \psi\left(t-\frac{x\sqrt{f_0 m_j}}{4Cz^2}\right) e^{-z^2} dz. \tag{2.7}$$

We determine the pressure distribution by substituting (2.7) in (1.10). In the case where a pressure or flow jump is assigned at the start of the tube, we put expressions (2.6) and (2.7) in the form

$$p(x, t)/p_0=\operatorname{erfc}\{[(3n+1)/4n]\xi\}; \tag{2.8}$$

$$M(x, t)/M_0=\operatorname{erfc}[(1/\sigma)\zeta], \tag{2.9}$$

where

$$\xi=x/2\sqrt{\varkappa_1 t};\quad \zeta=x/2\sqrt{\varkappa_2 t};\quad \varkappa_1=C^2/m_{10};\quad \varkappa_2=C^2/m_{20}.$$

Expressions (2.6)-(2.9) can be interpreted in two ways. On the one hand, they give the pressure and flow distributions in a viscoelastic tube at large times $Gt \gg \mu$. On the other hand, they give the pressure and flow distributions at an arbitrary time for infinitely small values of the time lag $\mu/G \to 0$, i.e., in an elastic tube.

Hence, at large times the viscoelastic properties of the tube material do not affect the motion. Figures 3 and 4 show the pressure and flow distributions calculated from Eqs. (2.8) and (2.9) for different values of the nonlinearity parameter n of "power-law" media and different values of the yield stress $\tau_0(\beta_0)$ and parameter n of nonlinear viscoplastic media; in the particular case where n = 1 (viscous liquid) the results agree completely with those in [3, 4].

LITERATURE CITED

1. R. M. Sattarov, "Hydraulic shock of power-law and nonlinear viscoplastic media in tubes of viscoelastic material," Zh. Prikl. Mekh. Tekh. Fiz., No. 3 (1975).
2. B. M. Smol'skii, Z. P. Shul'man, and V. M. Gorislavets, Rheodynamics and Heat Transfer of Nonlinear Viscoplastic Materials [in Russian], Nauka i Tekhnika, Minsk (1970).
3. I. P. Ginzburg, "Hydraulic shock in tubes of viscoelastic material," Vestn. Leningr. Univ., 3, No. 13, (1956).
4. I. A. Charnyi, Unsteady Motion of a Real Liquid in Tubes [in Russian], Nedra, Moscow (1975).